

PRESCRIBING THE PRESCHWARZIAN IN SEVERAL COMPLEX VARIABLES.

RODRIGO HERNÁNDEZ

ABSTRACT. We solve the several complex variables preSchwarzian operator equation $[Df(z)]^{-1}D^2f(z) = A(z)$, $z \in \mathbb{C}^n$, where $A(z)$ is a bilinear operator and f is a \mathbb{C}^n valued locally biholomorphic function on a domain in \mathbb{C}^n . Then one can define a several variables $f \rightarrow f_\alpha$ transform via the operator equation $[Df_\alpha(z)]^{-1}D^2f_\alpha(z) = \alpha[Df(z)]^{-1}D^2f(z)$, and thereby, study properties of f_α . This is a natural generalization of the one variable operator $f_\alpha(z)$ in [6] and the study of its univalence properties, e.g., the work of Royster [23] and many others. Möbius invariance and the multivariable Schwarzian derivative operator of T. Oda [17] play a central role in this work.

1. INTRODUCTION

Consider the class \mathcal{S} of functions f holomorphic and univalent in the disk $\mathbb{D} = \{z : |z| < 1\}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. Let $\alpha \in \mathbb{C}$, $f \in \mathcal{S}$ and define the integral transform

$$(1.1) \quad f_\alpha(z) = \int_0^z [f'(w)]^\alpha dw,$$

where the power is defined by the branch of the logarithm for which $\log f'(0) = 0$, [6]. A question considered in [6] is to determinate the values of α for which $f_\alpha \in \mathcal{S}$. In [23] Royster exhibited non-univalent mappings f_α for each complex $\alpha \neq 1$ with $|\alpha| > 1/3$. In fact, consider functions of the form

$$(1.2) \quad f(z) = \exp(\mu \log(1 - z)),$$

which are univalent if and only if μ lies in one of the closed disks

$$|\mu + 1| \leq 1, \quad |\mu - 1| \leq 1.$$

Royster showed that for any such value of μ , the function in (1.1) is not univalent for each α with $|\alpha| > 1/3$ and $\alpha \neq 1$. Moreover Pfaltzgraff, using the Ahlfors univalence criterion [1], proved that for any $f \in \mathcal{S}$, if $|\alpha| \leq 1/4$ then f_α is univalent

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in \mathbb{D} , see [19].

Let f be a locally univalent mapping in \mathbb{D} and f_α defined by equation (1.1) then $f'_\alpha(z) = [f'(z)]^\alpha$, which implies that

$$\frac{f''_\alpha}{f'_\alpha}(z) = \alpha \frac{f''}{f'}(z).$$

If f and g satisfy that $g''/g'(z) = f''/f'(z)$ then $\log(g'(z)) = \log(f'(z))$ when $f'(0) = g'(0)$. Therefore $g = f$ if $f(0) = g(0)$. Thus

$$(1.3) \quad f_\alpha(z) = \int_0^z [f'(w)]^\alpha dw \Leftrightarrow \frac{f''_\alpha}{f'_\alpha}(z) = \alpha \frac{f''}{f'}(z).$$

This equivalence in one variable suggests our idea to define the several variables generalization of f_α via operator equation

$$(1.4) \quad [Df_\alpha(z)]^{-1} D^2 f_\alpha(z)(\cdot, \cdot) = \alpha [Df(z)]^{-1} D^2 f(z)(\cdot, \cdot).$$

Yoshida developed, [25], a complete description of prescribing Oda's Schwarzian derivatives [17] in terms of a completely integrable system of differential equations. The description involves operators $S_{ij}^k f$ and $S_{ij}^0 f$ of orders two and three respectively, coefficients of the system and Möbius invariants. In fact, the $S_{ij}^k f$ operators are the operator of least order that vanish for Möbius mappings. This is a strong difference with one complex variable where the third order Schwarzian operator is the lowest order operator annihilated by all Möbius mappings. For $n = 1$, the Möbius group has dimension 3, which allows to set $f(z_0)$, $f'(z_0)$ and $f''(z_0)$ of a holomorphic mapping f at a given point z_0 arbitrarily. It would therefore be pointless to seek a Möbius invariant differential operator of order 2. But for $n > 1$ the number of parameters involved in the value and all derivatives of order 1 and 2 of a locally biholomorphic mapping is $n^2(n+1)/2 + n^2 + n$, and exceeds the dimension of the corresponding Möbius group in \mathbb{C}^n , which is $n^2 + 2n$. By the definition of the Schwarzian derivatives, we have that $S_{ij}^k F = S_{ji}^k F$ for all k and $\sum_{j=1}^n S_{ij}^j F = 0$ and we see there are exactly $n(n-1)(n+2)/2$ independent terms $S_{ij}^k F$, which is equal to the excess mentioned above.

A different approach to obtain the invariant operators S_{ij}^k , S_{ij}^0 has been developed by Molzon and Tamanoi ([14]). In addition, Molzon and Pinney had earlier developed equivalent invariant operators in the context of complex manifolds ([13]).

The operator

$$P_f(z) = [Df(z)]^{-1} D^2 f(z)(\cdot, \cdot)$$

introduced by Pfaltzgraff in [18] is the “natural” way to extend the classical one variable operator preSchwarzian f''/f' . Furthermore, the author in [18] extended the classical univalence criterion of Becker, ([2]), to several variables. The question now is how to extend the equation (1.1) to \mathbb{C}^n . It is necessary to understand

when one can recover the function f from a given P_f . We shall show a strong connection between this operator and the Schwarzian derivatives operator $SF(z)(\cdot, \cdot)$, introduced in [11]. Indeed, the problem of prescribing P_f can be reduced to understanding how to prescribe $S_{ij}^k f$ in terms of P_f . This is achieved via completely integrable system generated by $S_{ij}^k f$ and corresponding “new differential conditions” on the elements of P_f . We then use this theory to extend the classical single variable problem about the univalence of f_α by using equation (1.4) to define f_α in several complex variables.

2. ODA SCHWARZIAN AND MÖBIUS INVARIANTS

Let $f : \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a locally biholomorphic mapping defined on some domain Ω . T.Oda in [17] defined the Schwarzian derivatives of $f = (f_1, \dots, f_n)$ as

$$(2.1) \quad S_{ij}^k f = \sum_{l=1}^n \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1} \left(\delta_i^k \frac{\partial}{\partial z_j} + \delta_j^k \frac{\partial}{\partial z_i} \right) \log J_f,$$

where $i, j, k = 1, 2, \dots, n$, J_f is the jacobian determinant of the differential Df and δ_i^k are the Kronecker symbols. For $n > 1$ the Schwarzian derivatives have the following properties:

$$(2.2) \quad S_{ij}^k f = 0 \quad \text{for all } i, j, k = 1, 2, \dots, n \quad \text{iff} \quad f(z) = M(z),$$

for some Möbius transformation

$$M(z) = \left(\frac{l_1(z)}{l_0(z)}, \dots, \frac{l_n(z)}{l_0(z)} \right),$$

where $l_i(z) = a_{i0} + a_{i1}z_1 + \dots + a_{in}z_n$ with $\det(a_{ij}) \neq 0$. Furthermore, for a composition

$$(2.3) \quad S_{ij}^k(g \circ f)(z) = S_{ij}^k f(z) + \sum_{l,m,r=1}^n S_{lm}^r g(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r}, \quad w = f(z).$$

From this chain rule it can be shown that $S_{ij}^k f = S_{ij}^k g$ for all $i, j, k = 1, \dots, n$ if and only if $g = T \circ f$ for some Möbius transformation. The $S_{ij}^0 f$ coefficients are given by

$$S_{ij}^0 f(z) = J_f^{1/(n+1)} \left(\frac{\partial^2}{\partial z_i \partial z_j} J_f^{-1/(n+1)} - \sum_{k=1}^n \frac{\partial}{\partial z_k} J_f^{-1/(n+1)} S_{ij}^k f(z) \right).$$

In his work, Oda gives a description of the functions with prescribed Schwarzian derivatives $S_{ij}^k f$ ([17]). Consider the following overdetermined system of partial differential equations,

$$(2.4) \quad \frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^n P_{ij}^k(z) \frac{\partial u}{\partial z_k} + P_{ij}^0(z) u, \quad i, j = 1, 2, \dots, n,$$

where $z = (z_1, z_2, \dots, z_n) \in \Omega \subset \mathbb{C}^n$ and $P_{ij}^k(z)$ are holomorphic functions for $i, j, k = 0, \dots, n$. The system (2.4) is called *completely integrable* if there are at most $n+1$ linearly independent solutions, and is said to be in *canonical form* (see [24]) if the coefficients satisfy

$$\sum_{j=1}^n P_{ij}^j(z) = 0, \quad i = 1, 2, \dots, n.$$

T. Oda proved that (2.4) is a completely integrable system in canonical form if and only if $P_{ij}^k = S_{ij}^k f$ for a locally biholomorphic mapping $f = (f_1, \dots, f_n)$, where $f_i = u_i/u_0$ for $1 \leq i \leq n$ and u_0, u_1, \dots, u_n is a set of linearly independent solutions of the system. For a given mapping f , $u = (J_f)^{-\frac{1}{n+1}}$ is always a solution of (2.4) with $P_{ij}^k = S_{ij}^k f$.

Definition 2.1. We define the *Schwarzian derivative operator* as the operator $S_f(z) : T_z\Omega \rightarrow T_{f(z)}\Omega$ given by

$$S_f(z)(\vec{v}, \vec{w}) = (\vec{v}^t S^1 f(z) \vec{w}, \dots, \vec{v}^t S^n f(z) \vec{w}),$$

where $S^k f$ is the $n \times n$ matrix defined by $(S_{ij}^k f)_{ij}$ and $\vec{v} \in T_z\Omega$.

The Schwarzian derivative operator [12] can be rewritten as

$$S_f(z)(\vec{v}, \vec{w}) = [Df(z)]^{-1} D^2 f(z)(\vec{v}, \vec{w}) - \frac{1}{n+1} (\nabla \log J_f(z) \cdot \vec{v}) \vec{w} - \frac{1}{n+1} (\nabla \log J_f(z) \cdot \vec{w}) \vec{v},$$

and the system (2.4) as

$$(2.6) \quad \text{Hess } u(z)(\cdot, \cdot) = \nabla u(z) \cdot S_f(z)(\cdot, \cdot) + S_f^0(z)(\cdot, \cdot) u(z),$$

where S_f^0 is a $n \times n$ matrix defined by $(S_{ij}^0 f)_{ij}$. We include in this section two lemmas that complement the work of Oda.

Lemma 2.2. *Let $f : \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a locally biholomorphic mapping and $u_0 = J_f^{-1/n+1}$, then*

$$f = \frac{\vec{u}}{u_0} = \left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0} \right),$$

where u_0, u_1, \dots, u_n are linearly independent solutions of (2.4)

Proof. We will prove that $\vec{u} = f u_0$ is solution of the equation (2.6). It follows that $Df u_0 + f \nabla u_0 = Du$, from where

$$D^2 f \cdot u_0 + 2Df \cdot \nabla u_0 + f \cdot \text{Hess } u_0 = D^2 u.$$

Using the system we have that

$$D^2 f \cdot u_0 + 2Df \cdot \nabla u_0 - Df \cdot u_0 \cdot S_f + Du \cdot S_f + S_f^0 \cdot u = D^2 u.$$

Considering the equation (2.5) with $u_0 = J_f^{-1/n+1}$ we have that

$$D^2 f \cdot u_0 + 2Df \cdot \nabla u_0 - Df \cdot Sf \cdot u_0 = 0,$$

and $D^2 u(\cdot, \cdot) = Du(S_f(\cdot, \cdot)) + S_f^0(\cdot, \cdot)u$, hence u_i with $i = 1, \dots, n$ and u_0 are independent solutions of the system (2.4). \square

Lemma 2.3. *Let u_0 be a solution of the system (2.4). Then there exists a function $f = \vec{u}/u_0$ where $\vec{u} = (u_1, \dots, u_n)$ and u_i with $i = 0, 1, \dots, n$ are independent solutions of the system (2.4) where $u_0 = J_f^{-1/n+1}$. The function f will be holomorphic away from the zero set of u_0 .*

Proof. According to previous lemma we can find $F = \vec{v}/v_0$ where $\{v_0, v_1, \dots, v_n\}$ are a linearly independent solutions of the system (2.4) with $P_{ij}^k = S_{ij}^k$ and $v_0 = J_F^{-1/n+1}$. As u_0 is solution of the system we have that $u_0 = \alpha_0 v_0 + \dots + \alpha_n v_n$. We need to find a Möbius mapping T such that

$$T \circ F = \left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0} \right) = f,$$

and $J_{T \circ F}^{-1/n+1} = u_0$. We have

$$\begin{aligned} J_{T \circ F}^{-1/n+1}(z) &= J_T^{-1/n+1}(F(z)) J_F^{-1/n+1}(z) \\ &= (\lambda_0 + \lambda_1 F_1(z) + \dots + \lambda_n F_n(z)) J_F^{-1/n+1}(z) \\ &= \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_n v_n, \end{aligned}$$

which will be equal to u_0 if we choose $\lambda_i = \alpha_i$ for all $i = 0, 1, \dots, n$. \square

3. RESULTS

Let $\Omega \subset \mathbb{C}^n$ be domain.

Theorem 3.1. *Let $f : \Omega \rightarrow \mathbb{C}^n$ be a locally biholomorphic mapping. The following statements are equivalent:*

- (i) $S_{ij}^0 f(z) \equiv 0$.
- (ii) *There exists a locally biholomorphic mapping $g : \Omega \rightarrow \mathbb{C}^n$ with $S_g = S_f$ and J_g constant.*
- (iii) *There exists a locally biholomorphic mapping $h : \Omega \rightarrow \mathbb{C}^n$ such that $S_h = S_f$ and $J_h^{-1/n+1} = 1/L(h)$, where $L(w) = \alpha_0 + \alpha_1 w_1 + \dots + \alpha_n w_n$.*

(iv) *Locally there exists a biholomorphic change of variables such that the system (2.4) with $P_{ij}^k = S_{ij}^k f$ reduces to $\text{Hess}(u) = 0$.*

Proof. (i) \rightarrow (ii). As $S_{ij}^0 f \equiv 0$, the system (2.4) reduces to

$$u_{ij} = \sum_{k=1}^n S_{ij}^k u_k.$$

Therefore $u \equiv c$ is solution, thus by Lemma (2.3) there exists a function g such that $J_g \equiv C$.

(ii) \rightarrow (iii). Let $g = T \circ h$ for some Möbius T to be determined. Then $J_g^{-1/n+1}(z) = J_T^{-1/n+1}(h(z))J_h^{-1/n+1}(z)$. Since $J_g^{-1/n+1} \equiv C$ we have that

$$C = (a_0 + a_1 h_1 + \cdots + a_n h_n) J_h^{-1/n+1}(z),$$

from where the result obtains after scaling h .

(iii) \rightarrow (iv). Suppose h has $J_h^{-1/n+1} = 1/L(h)$. The previous argument shows that by choosing T appropriately, we can produce $g = T \circ h$ with $J_g \equiv 1$. Hence $S_g(z)(\cdot, \cdot) = (Dg(z))^{-1} D^2 g(z)(\cdot, \cdot)$, and the system (2.4) reduces to

$$\text{Hess } u(z)(\cdot, \cdot) = \nabla u(z) \cdot S_g(z)(\cdot, \cdot).$$

We consider $D(\nabla u(z)(Dg(z))^{-1})(\cdot, \cdot)$:

$$\begin{aligned} D(\nabla u(z)(Dg(z))^{-1})(\cdot, \cdot) &= \text{Hess } u(z)((Dg(z))^{-1}(\cdot, \cdot) \\ &\quad - \nabla u(z) \cdot (Dg(z))^{-1} D^2 g(z)((Dg(z))^{-1}(\cdot, \cdot)) \\ &= \nabla u(z) \cdot S_g(z)((Dg(z))^{-1}(\cdot, \cdot) \\ &\quad - \nabla u(z) \cdot (Dg(z))^{-1} D^2 g(z)((Dg(z))^{-1}(\cdot, \cdot)) = 0. \end{aligned}$$

Let φ a local inverse of g . Therefore $U(w) = u(\varphi(w))$ satisfies that $\nabla U = \nabla u \cdot D\varphi = \nabla u(z)(Dg(z))^{-1}$, thus $\text{Hess } U(w) \equiv 0$.

(iv) \rightarrow (i). Since $\text{Hess } u(s) \equiv 0$, then $u \equiv c$ is a solution of this system (2.4) therefore $S_{ij}^0 f \equiv 0$. \square

Theorem 3.2. *Let $f : \Omega \rightarrow \mathbb{C}^n$ be a locally biholomorphic mapping. There exists a function $g : \Omega \rightarrow \mathbb{C}^n$ locally biholomorphic such that*

$$(3.1) \quad Dg(z) = Df(z) J_f^{-\frac{2}{n+1}}$$

if and only if $S_{ij}^0 f \equiv 0$ for all i and j . The function g will have $S_g = S_f$.

Proof. Suppose (3.1) holds. A straightforward calculation shows that

$$(Dg(z))^{-1}D^2g(z)(v, v) = S_f(z)(v, v).$$

The coordinate functions g^i of function g satisfy

$$dg^i = J_f^{-2/n+1}df^i.$$

Since $0 = d^2g^i = d^2f^i$ we conclude that J_f must be a constant. By Theorem 3.1 we conclude that $S_{ij}^0f \equiv 0$ for all i and j . Reciprocally, if $S_{ij}^0f \equiv 0$ then there exists a constant solution of the system (2.4), and by Lemma 2.2 there exists a mapping g with $S_g = S_f$ and $J_g^{-1/n+1} \equiv C$. By (2.5), $S_g = P_g = S_f$. \square

Remark 3.3. Considering $S_{ij}^0f \equiv 0$ then $cDf = Dg$ for some constant c . When $c = J_f^{-2/n+1}$ we have that

$$P_g(z) = S_f(z) = P_f(z).$$

M.A. Goldberg in [7] showed that, in terms of the our operator,

$$(3.2) \quad \text{tr} \{Df(z)^{-1}D^2f(z)(\vec{v}_i, \cdot)\} = \frac{\partial}{\partial z_i} \log J_f(z),$$

where $\vec{v}_i = (0, \dots, 1, \dots, 0)$ with 1 is in position i . We use this result to prove the next theorem of uniqueness.

Theorem 3.4. *Let f, g be locally biholomorphic mappings defined in Ω . Then $P_f(z)(\cdot, \cdot) = P_g(z)(\cdot, \cdot)$ if and only if $f = T \circ g$, where $T(z) = Az + b$ with A is a $n \times n$ constant matrix and $b \in \mathbb{C}^n$.*

Proof. Let f and g be locally biholomorphic mappings in Ω . As $P_f(z)(\vec{v}_i, \cdot) = P_g(z)(\vec{v}_i, \cdot)$ for all $i = 1, \dots, n$ then by equation (3.2) we have that

$$(3.3) \quad \nabla \log J_f(z) = \nabla \log J_g(z).$$

Using equation (2.5) we can conclude that $S_f(z) = S_g(z)$ for all z . Hence $g = T \circ f$ for some Mobius mapping T . But $\log J_g(z) = \log J_T(f(z)) + \log J_f(z)$ and equation (3.3) we have that $\log J_T(z)$ is a constant, therefore $T(z) = Az + b$ for some $n \times n$ matrix A and $b \in \mathbb{C}^n$. Reciprocally, if $f = T \circ g$ with $T(z) = Az + b$ for some $n \times n$ matrix A and $b \in \mathbb{C}^n$, is easy to see that $Df(z) = DT(f(z))Dg(z) = ADf(z)$, which implies that $P_f(z) = P_g(z)$. \square

Theorem 3.5. *Let $A(z)$ be a bilinear operator defined in Ω by*

$$A(z)(\vec{v}, \cdot) = \begin{pmatrix} a_{11}^1 v_1 + \dots + a_{1n}^1 v_n & \cdot & \cdot & \cdot & \cdot & a_{n1}^1 v_1 + \dots + a_{nn}^1 v_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{11}^n v_1 + \dots + a_{1n}^n v_n & \cdot & \cdot & \cdot & \cdot & a_{n1}^n v_1 + \dots + a_{nn}^n v_n \end{pmatrix}$$

where $a_{ij}^k = a_{ij}^k(z)$ and $\vec{v} = (v_1, \dots, v_n)$. Then there exists a function $f : \Omega \rightarrow \mathbb{C}^n$ locally biholomorphic such that $P_f(z) = A(z)$ if and only if the following statements hold:

(i) $a_{ij}^k(z) = a_{ji}^k(z)$ for all $i, j, k = 1, \dots, n$;

(ii) there exists a holomorphic function $\varphi : \Omega \rightarrow \mathbb{C}$ such that

$$a_{1j}^1(z) + a_{2j}^2(z) + \dots + a_{nj}^n(z) = \frac{\partial \varphi}{\partial z_j}(z) \quad \forall j = 1, \dots, n;$$

(iii) $\exp(-\frac{\varphi}{n+1})$ is a solution of the system (2.4) with $P_{ij}^k(z)$ given by

$$P_{ij}^k(z) = a_{ij}^k(z) - \frac{1}{n+1} (\delta_i^k \operatorname{tr} \{A(z)(\vec{v}_j, \cdot) + \delta_j^k \operatorname{tr} \{A(z)(\vec{v}_i, \cdot)\}) \quad i, j, k = 1, \dots, n,$$

and $P_{ij}^0(z)$ are defined in terms of $P_{ij}^k(z)$ such that the integrable condition of the system ([25] page 129-130) holds.

Proof. Using (i) and (ii) we have that

$$\operatorname{tr} \{A(z)(\lambda, \cdot)\} = \nabla \varphi(z) \cdot \lambda.$$

For given $A(z)$ we can construct a bilinear mapping $\Lambda(z)(\lambda, \mu)$ as

$$\Lambda(z)(\lambda, \mu) = A(z)(\lambda, \mu) - \frac{1}{n+1} \operatorname{tr} \{A(z)(\lambda, \cdot)\} \mu - \frac{1}{n+1} \operatorname{tr} \{A(z)(\mu, \cdot)\} \lambda.$$

Each component of $\Lambda(z)$ is P_{ij}^k defined by

$$a_{ij}^k(z) - \frac{1}{n+1} (\delta_i^k \operatorname{tr} \{A(z)(\vec{v}_j, \cdot) + \delta_j^k \operatorname{tr} \{A(z)(\vec{v}_i, \cdot)\}) .$$

These coefficients satisfy $\sum_i P_{ik}^k = 0$ for all $k = 1, \dots, n$. Now we define coefficients P_{ij}^0 in terms of P_{ij}^k with $k = 1, \dots, n$ such that the integrability conditions in [25] hold, (see pages 129-130). Thus, the system (2.4) is completely integrable and in canonical form. Hence we can construct a function f such that $S_f(z) = \Lambda(z)$. By (iii) we have that

$$J_f^{-1/n+1} = \exp(-\frac{\varphi}{n+1}).$$

As S_f is defined by equation (2.5) we conclude that

$$\operatorname{tr} \{A(z)(\lambda, \cdot)\} = \frac{1}{n+1} \nabla J_f(z) \cdot \lambda,$$

which implies that

$$P_f(z) = (Df(z))^{-1} D^2 f(z)(\cdot, \cdot) = A(z)(\cdot, \cdot).$$

Reciprocally, is easy to see that $P_f(z)$ satisfies (i), (ii) and (iii). \square

Observe that $\alpha[Df(z)]^{-1}D^2f(z)(\vec{v}, \cdot)$ by locally biholomorphic function f , satisfies (i), (ii) and (iii) of the Theorem 3.4.

Definition 3.6. Let f be a locally biholomorphic mapping in Ω such that $f(0) = 0$ and $Df(0) = Id$. We define f_α in Ω as the locally biholomorphic mapping for which

$$(3.4) \quad [Df_\alpha(z)]^{-1}D^2f_\alpha(z)(\cdot, \cdot) = \alpha[Df(z)]^{-1}D^2f(z)(\cdot, \cdot),$$

and $f_\alpha(0) = 0$, $Df_\alpha(0) = Id$.

As a generalization of the problem raised in [6], one can ask the question of determining the values of α for which the mapping f_α is univalent when f is univalent or even just locally univalent. A partial answer is given below when f is convex in the unit ball \mathbb{B}^n . Theorem 3.5 shows another partial result for compact linear invariant families. Since the class of univalent mappings in \mathbb{B}^n fails to be compact ($n > 1$), we think it is unlikely that there exists an $\alpha_0 > 0$ small enough so that f_α is univalent for any $|\alpha| \leq \alpha_0$ and f univalent in \mathbb{B}^n . An interesting compact family of univalent mappings to consider would be the class S_0 of univalent mappings in \mathbb{B}^n that have a parametric representation.

Example 3.7. Let $f(z_1, z_2) = (\phi_\alpha(z_1), \psi_\alpha(z_2))$ be a locally univalent mapping defined in \mathbb{B}^2 such that $\phi_\alpha(z_1)$ and $\psi_\alpha(z_2)$ are defined by the equation (1.1) where ϕ and ψ are locally univalent analytic mappings defined in the unit disc such that $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0) = 1$ and suppose that $z = (z_1, z_2) \in \mathbb{B}^2$. Its Schwarzian derivatives satisfy

$$\begin{aligned} S_{11}^1 f(z_1, z_2) &= \frac{\phi_\alpha''}{\phi_\alpha'}(z_1) = \alpha \frac{\phi''}{\phi'}(z_1), \\ S_{22}^2 f(z_1, z_2) &= \frac{\psi_\alpha''}{\psi_\alpha'}(z_2) = \alpha \frac{\psi''}{\psi'}(z_2), \\ S_{22}^1 f(z_1, z_2) &= S_{11}^2 f(z_1, z_2) = 0. \end{aligned}$$

Now, let $f(z) = (\psi(z_1), \phi(z_2))$. Then the corresponding mapping f_α has the property that its Schwarzian derivatives are

$$\begin{aligned} S_{11}^1 f_\alpha(z_1, z_2) &= \alpha S_{11}^1 f(z_1, z_2) = \alpha \frac{\phi''}{\phi'}(z_1), \\ S_{22}^2 f_\alpha(z_1, z_2) &= \alpha S_{22}^2 f(z_1, z_2) = \alpha \frac{\psi''}{\psi'}(z_2), \\ S_{22}^1 f_\alpha(z_1, z_2) &= S_{11}^2 f_\alpha(z_1, z_2) = 0. \end{aligned}$$

Therefore $S_{ij}^k f = S_{ij}^k f_\alpha$ which implies that there exists a Möbius mapping M such that $M \circ f = f_\alpha$. But $f(0) = 0 = f_\alpha(0)$, $DF(0) = Id = Df_\alpha(0)$ and $\nabla \log J_f = \nabla \log J_{f_\alpha} = \alpha \nabla \log J_f$, then $f = f_\alpha$. Thus

$$f(z) = (\phi(z_1), \psi(z_2)) \implies f_\alpha(z) = (\phi_\alpha(z_1), \psi_\alpha(z_2)),$$

where ϕ_α and ψ_α are defined by (1.1). By the way, in this example if $|\alpha| < 1/4$ then f_α will be univalent in \mathbb{B}^2 . Moreover if $\phi(z_1)$ is a univalent mapping defined by (1.2) and $\psi(z_2) = z_2$, then the mapping $f(z) = (\phi(z_1), \psi(z_2))$ is univalent and the corresponding mapping f_α is not univalent if $|\alpha| > 1/3$ and $\alpha \neq 1$.

In [9] the author proved that $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$ locally biholomorphic mapping is convex if and only if $1 - \operatorname{Re}\langle [Df(z)]^{-1} D^2 f(z)(u, u), z \rangle > 0$ for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$. Thus, if $0 \leq \alpha \leq 1$ then f_α is a convex mapping when f is a convex mapping since

(3.5)

$$1 - \operatorname{Re}\langle [Df_\alpha(z)]^{-1} D^2 f_\alpha(z)(u, u), z \rangle = 1 - \alpha \operatorname{Re}\langle [Df(z)]^{-1} D^2 f(z)(u, u), z \rangle > 0.$$

Example 3.8. Let f be a univalent function in \mathbb{D} . We consider the Roper-Suffridge extension (see[21]) to \mathbb{B}^2 of f to the function

$$\Phi_f(z) = \left(f(z_1), \sqrt{f'(z_1)} z_2 \right).$$

Thus,

$$[D\Phi_f(z)]^{-1} [D^2 \Phi_f(z)](\vec{v}, \cdot) = \begin{pmatrix} \frac{f''}{f'}(z_1) v_1 & 0 \\ \frac{1}{2} z_2 S f(z_1) v_1 + \frac{1}{2} \frac{f''}{f'}(z_1) v_2 & \frac{1}{2} \frac{f''}{f'}(z_1) v_1 \end{pmatrix}.$$

A straightforward calculation shows that

$$(\Phi_f)_\alpha(z) = \left(f_\alpha(z_1), z_2 \sqrt{f'_\alpha(z_1)} + y(z_1) \right),$$

where f_α is defined by equation (1.1) and y satisfies that

$$y'' - \alpha \frac{f''}{f'} y' = \frac{\alpha(\alpha - 1)}{4} \left(\frac{f''}{f'} \right)^2 (f')^{\alpha/2}.$$

Moreover Φ_f is univalent when f is univalent, in fact if f is convex then Φ_f is convex. On the other hand, $(\Phi_f)_\alpha$ is univalent if f_α is univalent which holds for $|\alpha| \leq 1/4$ for all univalent mappings f .

Theorem 3.9. *Let $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$ be a locally biholomorphic mapping such that the norm order of the linear invariant family generated by f is $\beta < \infty$. Then f_α is univalent if $|\alpha| \leq \frac{1}{2\beta + 1}$.*

Proof. Let ϕ be a automorphism of \mathbb{B}^n such that $\phi(0) = \zeta$. The mapping $g(z) = D\phi(0)^{-1} Df(\phi(0))^{-1} (f(\phi(z)) - f(\phi(0)))$ belongs to the family generated by f , therefore $\|D^2 g(0)\| \leq \beta$. But

$$\begin{aligned} D^2 g(0)(\cdot, \cdot) &= D\phi(0)^{-1} Df(\zeta)^{-1} Df(w) D^2 \phi(0)(\cdot, \cdot) + \\ &\quad D\phi(0)^{-1} Df(\zeta)^{-1} D^2 f(\zeta) (D\phi(0)(\cdot), D\phi(0)(\cdot)). \end{aligned}$$

Evaluating in $D\phi(0)^{-1}(\zeta) = \zeta/(1 - \|\zeta\|^2)$, multiplication by α and using (3.4) we have that

$$\begin{aligned} \alpha D^2 g(0)(\zeta, \cdot) &= \alpha D\phi(0)^{-1} Df(\zeta)^{-1} Df(\zeta) D^2 \phi(0)(\zeta, \cdot) + \\ &\quad (1 - \|\zeta\|^2) D\phi(0)^{-1} Df_\alpha(\zeta)^{-1} D^2 f_\alpha(\zeta)(\zeta, D\phi(0)(\cdot)), \end{aligned}$$

where $D\phi(0)^{-1} D^2 \phi(\zeta, \cdot) = -\|\zeta\|^2(\cdot) - \zeta \zeta^*(\cdot)$. Thus, for all vectors $v = D\phi(0)^{-1}(u)$ it follows that

$$\begin{aligned} (1 - \|\zeta\|^2) Df_\alpha(\zeta)^{-1} D^2 f_\alpha(\zeta)(\zeta, u) &= \alpha D\phi(0) D^2 g(0)(\zeta, v) - \alpha \|\zeta\|^2 u \\ &\quad - \alpha (1 - \|\zeta\|^2) \zeta \zeta^* v. \end{aligned}$$

Then taking supremum over all vectors u with norm $\|u\| = 1$, we have that

$$\|(1 - \|\zeta\|^2) Df_\alpha(\zeta)^{-1} D^2 f_\alpha(\zeta)(\zeta, \cdot) + \alpha \|\zeta\|^2 I\| \leq |\alpha| (2\beta + 1),$$

hence by the generalization of Ahlfors and Becker result (see page 350 in [10]) we conclude that f_α satisfies the hypothesis of this theorem so is univalent in \mathbb{B}^n . \square

The last corollary is an immediate consequence.

Corollary 3.10. *Let \mathcal{F} be a linearly invariant family of locally biholomorphic mappings defined in \mathbb{B}^n of finite order β . Then f_α is univalent in \mathbb{B}^n for every $f \in \mathcal{F}$ and $|\alpha| \leq \frac{1}{2\beta + 1}$.*

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Facultad de Ciencias y Tecnología, Universidad Adolfo Ibáñez, Av. Balmaceda
1625 Recreo, Viña del Mar, Chile, rodrigo.hernandez@uai.cl